Modular forms, modular symbols

(PARI-GP version 2.15.2)

Modular Forms

Dirichlet characters

Characters are encoded in three different ways:

• a `t_INT` $D \equiv 0, 1 \bmod 4$: the quadratic character $(D/\cdot)$;

• a `t_INTMOD` $\texttt{Mod}(m, q)$, $m \in (\mathbf{Z}/q)^*$ using a canonical bijection with the dual group (the Conrey character $\chi_q(m, \cdot)$);

• a pair $[G, chi]$, where $G = znstar(q, 1)$ encodes $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \le k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$ and the vector $chi = [c_1, \ldots, c_k]$ encodes the character such that $\chi(g_j) = e(c_j/d_j)$.

initialize $G = (\mathbf{Z}/q\mathbf{Z})^*$	$\texttt{G = znstar}(q, 1)$
convert datum D to $[G, \chi]$	$\texttt{znchar}(D)$
Galois orbits of Dirichlet characters	$\texttt{chargalois}(G)$

Spaces of modular forms

Arguments of the form $[N, k, \chi]$ give the level weight and nebentypus χ; χ can be omitted: $[N, k]$ means trivial χ.

initialize $S_k^{\text{new}}(\Gamma_0(N), \chi)$	$\texttt{mfinit}([N, k, \chi], 0)$
initialize $S_k(\Gamma_0(N), \chi)$	$\texttt{mfinit}([N, k, \chi], 1)$
initialize $S_k^{\text{old}}(\Gamma_0(N), \chi)$	$\texttt{mfinit}([N, k, \chi], 2)$
initialize $E_k(\Gamma_0(N), \chi)$	$\texttt{mfinit}([N, k, \chi], 3)$
initialize $M_k(\Gamma_0(N), \chi)$	$\texttt{mfinit}([N, k, \chi])$
find eigenforms	$\texttt{mfsplit}(M)$
statistics on self-growing caches	$\texttt{getcache}()$

We let $M = \texttt{mfinit}(\ldots)$ denote a modular space.

describe the space M	$\texttt{mfdescribe}(M)$
recover (N, k, χ)	$\texttt{mfparams}(M)$
... the space identifier (0 to 4)	$\texttt{mfspace}(M)$
... the dimension of M over $\mathbf{C}$	$\texttt{mfdim}(M)$
... a $\mathbf{C}$-basis (f_i) of M	$\texttt{mfbasis}(M)$
... a basis (F_j) of eigenforms	$\texttt{mfeigenbasis}(M)$
... polynomials defining $\mathbf{Q}(\chi)(F_j)/\mathbf{Q}(\chi)$	$\texttt{mffields}(M)$

matrix of Hecke operator T_n on (f_i)	$\texttt{mfheckemat}(M, n)$
eigenvalues of w_Q	$\texttt{mfatkineigenvalues}(M, Q)$
basis of period poynomials for weight k	$\texttt{mfperiodpolbasis}(k)$
basis of the Kohnen $+$-space	$\texttt{mfkohnenbasis}(M)$
... new space and eigenforms	$\texttt{mfkohneneigenbasis}(M, b)$
isomorphism $S_k^+(4N, \chi) \to S_{2k-1}(N, \chi^2)$	$\texttt{mfkohnenbijection}(M)$

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of $S_k^{\text{new}}(\Gamma_0(N), \chi)$	$\texttt{mfdim}([N, k, \chi])$
dimension of $S_k(\Gamma_0(N), \chi)$	$\texttt{mfdim}([N, k, \chi], 1)$
dimension of $S_k^{\text{old}}(\Gamma_0(N), \chi)$	$\texttt{mfdim}([N, k, \chi], 2)$
dimension of $M_k(\Gamma_0(N), \chi)$	$\texttt{mfdim}([N, k, \chi], 3)$
dimension of $E_k(\Gamma_0(N), \chi)$	$\texttt{mfdim}([N, k, \chi], 4)$
Sturm's bound for $M_k(\Gamma_0(N), \chi)$	$\texttt{mfsturm}(N, k)$

$\Gamma_0(N)$ cosets

list of right $\Gamma_0(N)$ cosets	$\texttt{mfcosets}(N)$
identify coset a matrix belongs to	`mftocoset`

Cusps

a cusp is given by a rational number or `oo`.

lists of cusps of $\Gamma_0(N)$	$\texttt{mfcusps}(N)$
number of cusps of $\Gamma_0(N)$	$\texttt{mfnumcusps}(N)$
width of cusp c of $\Gamma_0(N)$	$\texttt{mfcuspwidth}(N, c)$
is cusp c regular for $M_k(\Gamma_0(N), \chi)$?	$\texttt{mfcuspisregular}([N, k, \chi], c)$

Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients	$\texttt{mftobasis(mf}, vec)$

There are also many predefined ones:

Eisenstein series E_k on $Sl_2(\mathbf{Z})$	$\texttt{mfEk}(k)$
Eisenstein-Hurwitz series on $\Gamma_0(4)$	$\texttt{mfEH}(k)$
unary θ function (for character ψ)	$\texttt{mfTheta}(\{\psi\})$
Ramanujan's Δ	$\texttt{mfDelta}()$
$E_k(\chi)$	$\texttt{mfeisenstein}(k, \chi)$
$E_k(\chi_1, \chi_2)$	$\texttt{mfeisenstein}(k, \chi_1, \chi_2)$
eta quotient $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$	$\texttt{mffrometaquo}(a)$
newform attached to ell. curve $E/\mathbf{Q}$	$\texttt{mffromell}(E)$
identify an L-function as a eigenform	$\texttt{mffromlfun}(L)$
θ function attached to $Q > 0$	$\texttt{mffromqf}(Q)$
trace form in $S_k^{\text{new}}(\Gamma_0(N), \chi)$	$\texttt{mftraceform}([N, k, \chi])$
trace form in $S_k(\Gamma_0(N), \chi)$	$\texttt{mftraceform}([N, k, \chi], 1)$

Operations on modular forms

In this section, f, g and the $F[i]$ are modular forms

$f \times g$	$\texttt{mfmul}(f, g)$
f/g	$\texttt{mfdiv}(f, g)$
f^n	$\texttt{mfpow}(f, n)$
$f(q)/q^v$	$\texttt{mfshift}(f, v)$
$\sum_{i \le k} \lambda_i F[i]$, $L = [\lambda_1, \ldots, \lambda_k]$	$\texttt{mflinear}(F, L)$
$f = g$?	`mfisequal(f,g)`
expanding operator $B_d(f)$	$\texttt{mfbd}(f, d)$
Hecke operator $T_n f$	$\texttt{mfhecke}(mf, f, n)$
initialize Atkin–Lehner operator w_Q	$\texttt{mfatkininit}(mf, Q)$
... apply w_Q to f	$\texttt{mfatkin}(w_Q, f)$
twist by the quadratic char $(D/\cdot)$	$\texttt{mftwist}(f, D)$
derivative wrt. $q \cdot d/dq$	$\texttt{mfderiv}(f)$
see f over an absolute field	$\texttt{mfreltoabs}(f)$
Serre derivative $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$	$\texttt{mfderivE2}(f)$
Rankin-Cohen bracket $[f, g]_n$	$\texttt{mfbracket}(f, g, n)$
Shimura lift of f for discriminant D	$\texttt{mfshimura}(mf, f, D)$

Properties of modular forms

In this section, $f = \sum_n f_n q^n$ is a modular form in some space M with parameters N, k, χ.

describe the form f	$\texttt{mfdescribe}(f)$
(N, k, χ) for form f	$\texttt{mfparams}(f)$
the space identifier (0 to 4) for f	$\texttt{mfspace}(mf, f)$
$[f_0, \ldots, f_n]$	$\texttt{mfcoefs}(f, n)$
f_n	$\texttt{mfcoef}(f, n)$
is f a CM form?	$\texttt{mfisCM}(f)$
is f an eta quotient?	$\texttt{mfisetaquo}(f)$
Galois rep. attached to all $(1, \chi)$ eigenforms	$\texttt{mfgaloistype}(M)$
... single eigenform	$\texttt{mfgaloistype}(M, F)$
... as a polynomial fixed by Ker ρ_F	$\texttt{mfgaloisprojrep}(M, F)$
decompose f on $\texttt{mfbasis}(M)$	$\texttt{mftobasis}(M, f)$
smallest level on which f is defined	$\texttt{mfconductor}(M, f)$
decompose f on $\oplus S_k^{\text{new}}(\Gamma_0(d))$, $d \mid N$	$\texttt{mftonew}(M, f)$
valuation of f at cusp c	$\texttt{mfcuspval}(M, f, c)$
expansion at ∞ of $f \mid_k \gamma$	$\texttt{mfslashexpansion}(M, f, \gamma, n)$
n-Taylor expansion of f at i	$\texttt{mftaylor}(f, n)$
all rational eigenforms matching criteria	`mfeigensearch`
... forms matching criteria	`mfsearch`

Forms embedded into C

Given a modular form f in $M_k(\Gamma_0(N), \chi)$ its field of definition $Q(f)$ has $n = [Q(f) : Q(\chi)]$ embeddings into the complex numbers. If $n = 1$, the following functions return a single answer, attached to the canonical embedding of f in $\mathbf{C}[[q]]$; else a vector of n results, corresponding to the n conjugates of f.

complex embeddings of $Q(f)$	$\texttt{mfembed}(f)$
... embed coefs of f	$\texttt{mfembed}(f, v)$
evaluate f at $\tau \in \mathcal{H}$	$\texttt{mfeval}(f, \tau)$
L-function attached to f	$\texttt{lfunmf}(mf, f)$
... eigenforms of new space M	$\texttt{lfunmf}(M)$

Periods and symbols

The functions in this section depend on $[Q(f) : Q(\chi)]$ as above.

initialize symbol fs attached to f	$\texttt{mfsymbol}(M, f)$
evaluate symbol fs on path p	$\texttt{mfsymboleval}(fs, p)$
Petersson product of f and g	$\texttt{mfpetersson}(fs, gs)$
period polynomial of form f	$\texttt{mfperiodpol}(M, fs)$
period polynomials for eigensymbol FS	$\texttt{mfmanin}(FS)$

Modular Symbols

Let $G = \Gamma_0(N)$, $V_k = \mathbf{Q}[X, Y]_{k-2}$, $L_k = \mathbf{Z}[X, Y]_{k-2}$ and $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$. An element of Δ is a *path* between cusps of $X_0(N)$ via the identification $[b] - [a] \to$ path from a to b, coded by the pair $[a, b]$ where a, b are rationals or $\texttt{oo} = (1:0)$.

Let $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$; an element of $\mathbf{M}_k(G)$ is a V_k-valued *modular symbol*. There is a natural decomposition $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$ under the action of the $*$ involution, induced by complex conjugation. The `msinit` function computes either $\mathbf{M}_k$ $(\varepsilon = 0)$ or its $\pm$-parts $(\varepsilon = \pm 1)$ and fixes a minimal set of $\mathbf{Z}[G]$-generators (g_i) of Δ.

initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$	$\texttt{msinit}(N, k, \{\varepsilon = 0\})$
the level M	$\texttt{msgetlevel}(M)$
the weight k	$\texttt{msgetweight}(M)$
the sign ε	$\texttt{msgetsign}(M)$
Farey symbol attached to G	$\texttt{mspolygon}(M)$
... attached to $H < G$	$\texttt{msfarey}(F, inH)$
$H \backslash G$ and right G-action	$\texttt{mscosets}(genG, inH)$

$\mathbf{Z}[G]$-generators (g_i) and relations for Δ	$\texttt{mspathgens}(M)$
decompose $p = [a, b]$ on the (g_i)	$\texttt{mspathlog}(M, p)$

Create a symbol

Eisenstein symbol attached to cusp c	$\texttt{msfromcusp}(M, c)$
cuspidal symbol attached to $E/\mathbf{Q}$	$\texttt{msfromell}(E)$
symbol having given Hecke eigenvalues	$\texttt{msfromhecke}(M, v, \{H\})$
is s a symbol ?	$\texttt{msissymbol}(M, s)$

Operations on symbols

the list of all $s(g_i)$	$\texttt{mseval}(M, s)$
evaluate symbol s on path $p = [a, b]$	$\texttt{mseval}(M, s, p)$
Petersson product of s and t	$\texttt{mspetersson}(M, s, t)$

Operators on subspaces

An operator is given by a matrix of a fixed $\mathbf{Q}$-basis. H, if given, is a stable $\mathbf{Q}$-subspace of $\mathbf{M}_k(G)$: operator is restricted to H.

matrix of Hecke operator T_p or U_p	$\texttt{mshecke}(M, p, \{H\})$
matrix of Atkin-Lehner w_Q	$\texttt{msatkinlehner}(M, Q\{H\})$
matrix of the $*$ involution	$\texttt{msstar}(M, \{H\})$

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a $\mathbf{Q}$-basis. If H is a Hecke-stable subspace of $M_k(G)^+$ or $M_k(G)^-$, it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform $\sum_n a_n q^n$.

cuspidal subspace $S_k(G)^\varepsilon$	`mscuspidal`(M)
Eisenstein subspace $E_k(G)^\varepsilon$	`mseisenstein`(M)
new part of $S_k(G)^\varepsilon$	`msnew`(M)
split H into simple subspaces (of dim $\leq d$)	`mssplit`$(M, H, \{d\})$
dimension of a subspace	`msdim`(M)
$(a_1, \ldots, a_B)$ for attached newform	`msqexpansion`$(M, H, \{B\})$
$\mathbf{Z}$-structure from $H^1(G, L_k)$ on subspace A	`mslattice`(M, A)

Overconvergent symbols and p-adic L functions

Let M be a full modular symbol space given by `msinit` and p be a prime. To a classical modular symbol ϕ of level N ($v_p(N) \leq 1$), which is an eigenvector for T_p with nonzero eigenvalue a_p, we can attach a p-adic L-function L_p. The function L_p is defined on continuous characters of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$; in GP we allow characters $\langle\chi\rangle^{s_1}\tau^{s_2}$, where (s_1, s_2) are integers, τ is the Teichmüller character and χ is the cyclotomic character.

The symbol ϕ can be lifted to an *overconvergent* symbol Φ, taking values in spaces of p-adic distributions (represented in GP by a list of moments modulo p^n).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that $v_p(a_p) = 0$ if *flag* $= 0$ (fastest), and that $v_p(a_p) \geq$ *flag* otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions mu attached to Φ allowing to compute L_p to high accuracy.

initialize Mp to lift symbols	`mspadicinit`$(M, p, n, \{flag\})$
lift symbol ϕ	`mstooms`(Mp, ϕ)
eval overconvergent symbol Φ on path p	`msomseval`(Mp, Φ, p)
mu for p-adic L-functions	`mspadicmoments`$(Mp, S, \{D = 1\})$
$L_p^{(r)}(\chi^s)$, $s = [s_1, s_2]$	`mspadicL`$(mu, \{s = 0\}, \{r = 0\})$
$\hat{L}_p(\tau^i)(x)$	`mspadicseries`$(mu, \{i = 0\})$

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